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Unfriendly colorings of graphs with finite average degree

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Abstract

In an unfriendly coloring of a graph the color of every node mismatches that of the majority of its neighbors. We show that every probability measure preserving Borel graph with finite average degree admits a Borel unfriendly coloring almost everywhere. We also show that every bounded degree Borel graph of subexponential growth admits a Borel unfriendly coloring.

1. *Introduction*

Suppose that G is a locally finite graph on the vertex set X. We say that $c\colon X \to 2$ is an *unfriendly coloring* of G if for all $x \in X$ at least half of the neighbors of x receive a different color than x does. More formally, letting G_x denote the set of G-neighbors of x, such a function c is an unfriendly coloring if $|\{y \in G_x : c(x) \neq c(y)\}| \geqslant |\{y \in G_x : c(x) = c(y)\}|$. By a compactness argument unfriendly colorings exist for all locally finite graphs (see, for example, [1]). There exist graphs with uncountable vertex sets that have no unfriendly colorings [8]; it is not known if this is possible for graphs with countably many vertices.

A large and growing literature considers measure-theoretical analogues of classical combinatorial questions (see, for example, a survey by Kechris and Marks [6]). Following [3], we consider a measure-theoretical analogue of the question of unfriendly colorings. Suppose that G is a locally finite Borel graph on the standard Borel space X, and that μ is a Borel probability measure on X. We say that G is *μ-preserving* if there are countably many μ-preserving Borel involutions whose graphs cover the edges of G. Equivalently, G is μ-preserving if its connectedness relation E_G is a μ-preserving equivalence relation.

An important example of such graphs comes from probability measure preserving actions of finitely generated groups. Indeed let a group, generated by the finite symmetric set S, act by measure preserving transformations on a standard Borel probability space (X, μ). Then the associated graph $G = (X, E)$ whose edges are

$$E = \{(x, y) \,:\, y = sx \text{ for some } s \in S\}$$

is a μ-preserving graph.

In [3] it is shown that every free probability measure preserving action of a finitely generated group is weakly equivalent to another such action whose associated graph admits an unfriendly coloring. Note that such graphs are regular: (almost) every node has degree $|G_x| = |S|$. Recall that the (μ-)*cost* of a μ-preserving locally finite Borel graph G is simply half its average degree: $\mathrm{cost}(G) = \frac{1}{2}\int_X |G_x| \; d\mu$. Equivalently, using the Lusin–Novikov uniformization theorem (see, for example, [5, Lemma 18.12]) one may circumvent this factor of $\frac{1}{2}$ by instead computing $\int_X |\vec{G}_x| d\mu$, where $\vec{G}$ is an arbitrary measurable orientation of G.

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Our first result shows that every measure preserving graph with finite cost admits an (almost everywhere) unfriendly coloring.

THEOREM 1. *Suppose that (X, μ) is a standard probability space and that G is a μ-preserving locally finite Borel graph on X with finite cost. Then there is a μ-conull G-invariant Borel set A such that $G \upharpoonright A$ admits a Borel unfriendly coloring.*

We next explore how the invariance assumption can be weakened. Recall that a Borel probability measure is *G-quasi-invariant* if the G-saturation of every μ-null set remains μ-null. Such measures admit a *Radon–Nikodym* cocycle $\rho\colon G \to \mathbb{R}^+$ so that whenever $A \subseteq X$ is Borel and $f\colon A \to X$ a Borel partial injection whose graph is contained in G, then $\mu(f[A]) = \int_A \rho(x, f(x))\, d\mu$.

THEOREM 2. *Suppose that (X, μ) is a standard probability space, that G is a Borel graph on X with bounded degree d, and that μ is G-quasi-invariant, with corresponding Radon–Nikodym cocycle ρ. Suppose also that for all $(x, y) \in G$, $1 - \frac{1}{d} \leqslant \rho(x, y) \leqslant 1 + \frac{1}{d}$. Then there is a μ-conull G-invariant Borel set A such that $G \upharpoonright A$ admits a Borel unfriendly coloring.*

The proofs of Theorems 1 and 2 build on a potential function technique used in [**9**] (see also [**2**]) to study majority dynamics on infinite graphs; in the context of finite graphs, these techniques go back to Goles and Olivos [**4**]. Indeed, we show that in our settings (anti)-majority dynamics converge to an unfriendly coloring. The combinatorial nature of this technique allows us to extend our results to the Borel setting.

THEOREM 3. *Suppose that G is a bounded-degree Borel graph of subexponential growth. Then G admits a Borel unfriendly coloring.*

A natural question remains open: is there a locally finite Borel graph that does not admit a Borel unfriendly coloring? To the best of our knowledge this is not known, even with regards to the restricted class of bounded degree graphs. In contrast, Theorem 1 shows that for this class unfriendly colorings exist in the measure-preserving case. Still, we do not know if the finite cost assumption in Theorem 1 is necessary, or whether every locally finite measure preserving graph admits an almost everywhere unfriendly coloring.

2. *Proofs*

Proof of Theorem 1. By Kechris–Solecki–Todorcevic [**7**, Proposition 4.5], there exists a *repetitive sequence of independent sets*: a sequence $(X_n)_{n\in\mathbb{N}}$ of G-independent Borel sets so that each $x \in X$ is in infinitely many X_n. We will recursively build for each $n \in \mathbb{N}$ a Borel function $c_n\colon X \to 2$ which converge μ-almost everywhere to an unfriendly coloring of G.

The choice of c_0 is arbitrary, but we may as well declare it to be the constant 0 function.

Suppose now that c_n has been defined. We build c_{n+1} by 'flipping' the color of vertices in X_n with too many neighbors of the same color, and leaving everything else unchanged. More precisely, $c_{n+1}(x) = 1 - c_n(x)$ if $x \in X_n$ and $|\{y \in G_x : c_n(x) \neq c_n(y)\}| < |\{y \in G_x : c_n(x) = c_n(y)\}|$; otherwise, $c_{n+1}(x) = c_n(x)$.

To show that this sequence c_n converges μ-almost everywhere to an unfriendly coloring, we introduce some auxiliary graphs. Let G_n be the subgraph of G containing exactly those edges between vertices of the same c_n-color, so $x\ G_n\ y$ if and only if $x\ G\ y$ and $c_n(x) = c_n(y)$. Certainly for all $n \in \mathbb{N}$, $\operatorname{cost}(G_n) \leqslant \operatorname{cost}(G)$.

For $n \in \mathbb{N}$, let $B_n = \{x \in X : c_n(x) \neq c_{n+1}(x)\}$.

CLAIM. $\operatorname{cost}(G_n) - \operatorname{cost}(G_{n+1}) \geqslant \mu(B_n)$. □

Proof of the claim. Recall that, by the definition of c_{n+1}, $x \in B_n$ if and only if $x \in X_n$ and $|\{y \in G_x : c_n(x) \neq c_n(y)\}| < |\{y \in G_x : c_n(x) = c_n(y)\}|$. In particular, $B_n \subseteq X_n$ and hence is G-independent. Thus $G_{n+1} = G_n \triangle \{(x,y) : x \, G \, y \text{ and } \{x,y\} \cap B_n \neq \emptyset\}$. But for each $x \in B_n$, the above characterization of membership in B_n ensures that its G_{n+1}-degree is strictly smaller than its G_n-degree. The claim follows.

In particular, since the sum telescopes we see $\sum_{n \in \mathbb{N}} \mu(B_n) \leqslant \operatorname{cost}(G) < \infty$. Hence the set $C = \{x \in X : x \in B_n \text{ for infinitely many n}\}$ is μ-null by the Borel–Cantelli lemma. Let $A = X \setminus [C]_G$, so A is μ-conull.

CLAIM. c is an unfriendly coloring of $G \restriction A$.

Proof of the claim. Fix $x \in A$ and fix $k \in \mathbb{N}$ sufficiently large so that c_n has stabilized for x and all its (finitely many) neighbors beyond k. Fix $n > k$ so that $x \in X_n$. Since $c_n(x) = c_{n+1}(x)$, the definition of c_{n+1} implies $|\{y \in G_x : c_n(x) \neq c_n(y)\}| \geqslant |\{y \in G_x : c_n(x) = c_n(y)\}|$. But $c_n = c$ on $G_x \cup \{x\}$, and hence c is unfriendly as desired.

This completes the proof of the theorem. □

We next analyze the extent to which the measure-theoretic hypotheses may be weakened in this argument. Note that the sequence c_n of colorings is defined without using the measure at all (in fact it is determined by the graph G and the sequence (X_n) of independent sets); the measure only appears in the argument that sequence converges to a limit coloring. And even in this convergence argument, invariance only shows up in the critical estimate $\operatorname{cost}(G_n) - \operatorname{cost}(G_{n+1}) \geqslant \mu(B_n)$.

DEFINITION 4. Suppose that G is a locally finite Borel graph on standard Borel X, that $(X_n)_{n \in \mathbb{N}}$ is a sequence of G-independent Borel sets so that each $x \in X$ is in infinitely many X_n. We define the *flip sequence* $(c_n)_{n \in \mathbb{N}}$ of Borel functions from X to 2 as follows.

- c_0 is the constant 0 function.
- $c_{n+1}(x) = 1 - c_n(x)$ if $x \in X_n$ and $|\{y \in G_x : c_n(x) \neq c_n(y)\}| < |\{y \in G_x : c_n(x) = c_n(y)\}|$; otherwise, $c_{n+1}(x) = c_n(x)$.

DEFINITION 5. Given a locally finite Borel graph G on X and a sequence $(X_n)_{n \in \mathbb{N}}$ of repetitive independent sets as above, we say that a Borel measure μ on X is *compatible* with G and (X_n) if the corresponding flip sequence c_n converges on a μ-conull set.

The proof of Theorem 1 shows that whenever μ is a G-invariant Borel probability measure with respect to which the average degree of G is finite, then μ is compatible with every sequence of independent sets. We seek to weaken the invariance assumption when G has bounded degree.

PROPOSITION 6. *Suppose that G is a Borel graph on X with bounded degree d, and that μ is a G-quasi-invariant Borel probability measure with corresponding Radon–Nikodym cocycle ρ. Suppose further that for all $(x,y) \in G$, $1 - \frac{1}{d} \leqslant \rho(x,y) \leqslant 1 + \frac{1}{d}$. Then μ is compatible with every repetitive sequence of independent sets.*

Theorem 2 is an immediate consequence of this proposition.

Proof of Proposition 6. Put $\varepsilon = \frac{1}{d}$. Define a measure M on G by putting for all Borel $H \subseteq G$,

$$M(H) = \int_X |H_x| \, d\mu.$$

This new measure M will replace the occurrences of cost in the proof of Theorem 1.

Consider the flip sequence c_n, and define corresponding graphs $G_n \subseteq G$ by $x \, G_n \, y$ if and only if $x \, G \, y$ and $c_n(x) = c_n(y)$. As before, let B_n denote those $x \in X_n$ for which $c_{n+1}(x) \neq c_n(x)$. Note that the 'double counting' that occurred in the proof of Theorem 1 may no longer be true double counting, but the bound on ρ ensures that each edge is counted at most $(2+\varepsilon)$ times and at least $(2-\varepsilon)$ times.

Claim. $M(G_n) - M(G_{n+1}) \geqslant \mu(B_n)$.

Proof of the claim. Partition B_n into finitely many Borel sets $A_{r,s}$ where $x \in A_{r,s}$ if and only if x has r-many G_n neighbors and s-many G_{n+1} neighbors (so $r > s$ and $r + s \leqslant d$). We compute

$$\begin{aligned} M(G_n) - M(G_{n+1}) &= \int_X |(G_n)_x| - |(G_{n+1})_x| \, d\mu \\ &\geqslant \int_{B_n} (2-\varepsilon)|(G_n)_x| - (2+\varepsilon)|(G_{n+1})_x| \, d\mu \\ &= \sum_{r,s} \int_{A_{r,s}} (2-\varepsilon)r - (2+\varepsilon)s \, d\mu \\ &= \sum_{r,s} \int_{A_{r,s}} 2(r-s) - \varepsilon(r+s) \, d\mu \\ &\geqslant \sum_{r,s} \int_{A_{r,s}} 2 - d\varepsilon \, d\mu \\ &= \mu(B_n) \end{aligned}$$

as required.

The remainder of the argument is as in the proof of Theorem 1. □

Given Proposition 6, the proof of Theorem 3 is straightforward.

Proof of Theorem 3. Fix a degree bound d for G and put $\varepsilon = \frac{1}{d}$. It suffices to construct for each $x \in X$ a G-quasi-invariant Borel probability measure μ_x whose Radon–Nikodym cocycle is ε-bounded on G such that $\mu_x(\{x\}) > 0$. If we do so, Proposition 6 ensures that the flip sequence c_n converges μ_x-everywhere for each x, and thus it converges everywhere. The limit is then an unfriendly coloring by the same argument as in the final claim in the proof of Theorem 1.

To construct μ_x, first define a purely atomic measure ν_x supported on the G-component of x by declaring $\nu_x(\{y\}) = (1+\varepsilon)^{-\delta(x,y)}$, where δ denotes the graph metric. Subexponential growth of G ensures that $K = \sum_{y \in [x]_G} \nu_x(\{y\}) < \infty$. Finally, put $\mu_x = \frac{1}{K}\nu_x$. □

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